# The Joint Distribution For A Brownian Motion And Its Maximum And Minimum

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In Part I we defined $W_T$ to be the value of a Brownian motion at time T, $M_T^+$ to be the maximum value that the Brownian motion obtains over the time interval [0, $T$] and $M_T^-$ to be the minimum value that the Brownian motion obtains over the time interval [0, $T$]. Our goal is to find the joint distribution of $W_T$ and $M_T^+$ and the joint distribution of $W_T$ and $M_T^-$. Once we find the joint distributions we can add drift to $W_T$ and be one step closer to finding closed-form solutions to barrier options.

**Maximum Scenario:** The value of the Brownian motion hits or goes above barrier $m$ over the time interval $[0, T]$ and ends up below threshold $w$ at time $T$.

Given that $m \geq 0$ and $w \leq m$ we determined that for a Brownian motion with zero drift and variance $v$ the probability that the Brownian motion will increase to a point that is above the barrier $m$ before time T and end up below threshold $w$ at time T is...

$$Prob\Big[M_T^+ > m, W_T < w\Big] = \int\limits_{2m-w}^{\infty} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}z^2}\,\delta z \text{ ...where... } w \leq m \tag{1}$$

We can rewrite Equation (1) to be a function of its joint density function $f(x, y)$, which has yet to be determined. The rewritten equation is...

$$Prob\Big[M_T^+ > m, W_T < w\Big] = \int\limits_{m}^{\infty}\int\limits_{-\infty}^{w} f(x,y)\,\delta y\,\delta x \tag{2}$$

Since the left side of Equations (1) and (2) above are the same we can create the following equality...

$$\int\limits_{m}^{\infty}\int\limits_{-\infty}^{w} f(x,y)\,\delta y\,\delta x = \int\limits_{2m-w}^{\infty} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}z^2}\,\delta z \tag{3}$$

**Minimum Scenario:** The value of the Brownian motion hits or goes below barrier $m$ over the time interval $[0, T]$ and ends up above threshold $w$ at time $T$.

Given that $m \leq 0$ and $w \geq m$ we also determined that for a Brownian motion with zero drift and variance $v$ the probability that the Brownian motion will decrease to a point that is below the barrier $m$ before time T and end up above $w$ at time T is...

$$Prob\Big[M_T^- < m, W_T > w\Big] = \int\limits_{-\infty}^{2m-w} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}z^2}\,\delta z \text{ ...where... } w \geq m \tag{4}$$

We can rewrite Equation (4) to be a function of its joint density function $g(x, y)$, which is yet to be determined. The rewritten equation is...

$$Prob\Big[M_T^- < m, W_T > w\Big] = \int\limits_{-\infty}^{m}\int\limits_{w}^{\infty} g(x,y)\,\delta y\,\delta x \tag{5}$$

Since the left side of Equations (4) and (5) are the same we can create the following equality...

$$\int_{-\infty}^{m}\int_{w}^{\infty} g(x,y)\,\delta y\,\delta x = \int_{-\infty}^{2m-w} \frac{1}{\sqrt{2\pi v}}\,e^{-\frac{1}{2v}z^2}\delta z \tag{6}$$

## The Plan of Attack

Note that the joint density functions that we seek are $f(x,y)$, which is behind the double integral on the left hand side of Equation (3) above, and $g(x,y)$, which is behind the double integral on the left hand side of Equation (6) above. We can isolate these functions if we can get rid of the double integrals that contain them. We do this by taking derivatives of both the left and right hand side of Equations (3) and (6) with respect to the variables $m$ and $w$.

Before we move on to calculate the joint density functions we will make the following definitions...

$$\theta = 2m - w \;\; ...where... \;\; \frac{\delta\theta}{\delta m} = 2 \tag{7}$$

And...

$$\lambda = -\frac{1}{2v}(2m-w)^2 \;\; ...where... \;\; \frac{\delta\lambda}{\delta w} = 2 \times -\frac{1}{2v}(2m-w) \times -1 = \frac{2m-w}{v} \tag{8}$$

## The Joint Density Function of a Brownian Motion Without Drift and Its Maximum

We start by taking the derivative of the left side of Equation (3) with respect to $w$ which is...

$$\frac{\delta}{\delta w}\left(\int_{m}^{\infty}\int_{-\infty}^{w} f(x,y)\,\delta y\,\delta x\right) = \int_{m}^{\infty} f(x,w)\,\delta x \tag{9}$$

We then take the derivative of Equation (9) with respect to $m$ which is...

$$\frac{\delta}{\delta m}\left(\int_{m}^{\infty} f(x,w)\,\delta x\right) = -f(m,w) \tag{10}$$

We now move to the right side of Equation (3) and take its derivatives with respect to $m$ and $w$. Using the definition of theta in Equation (7) the right side of Equation (3) can be rewritten as...

$$F = \int_{\theta}^{\infty} \frac{1}{\sqrt{2\pi v}}\,e^{-\frac{1}{2v}z^2}\,\delta z \tag{11}$$

The derivative of Equation (11) with respect to $m$ is...

$$\frac{\delta F}{\delta\theta}\frac{\delta\theta}{\delta m} = -\frac{1}{\sqrt{2\pi v}}\,e^{-\frac{1}{2v}\theta^2} \times 2 = -\frac{2}{\sqrt{2\pi v}}\,e^{-\frac{1}{2v}(2m-w)^2} \tag{12}$$

We now take the derivative of Equation (12) with respect to $w$. Using the definition of lambda in Equation (8) Equation (12) can be rewritten as...

$$G = -\frac{2}{\sqrt{2\pi v}}\,e^{\lambda} \tag{13}$$

The derivative of Equation (13) with respect to $w$ is...

$$\frac{\delta G}{\delta\lambda}\frac{\delta\lambda}{\delta w} = -\frac{2}{\sqrt{2\pi v}}\,e^{\lambda} \times \frac{2m-w}{v} = -\frac{2(2m-w)}{v\sqrt{2\pi v}}\,e^{-\frac{1}{2v}(2m-w)^2} \tag{14}$$

Equation (10) and Equation (14) are the derivatives of the left and right side, respectively, of Equation (3) with respect to the variables $m$ and $w$. When we reverse the signs and equate these two equations the equation for the joint density of a Brownian motion with zero drift and its maximum is...

$$f(m,w) = \frac{2(2m-w)}{v\sqrt{2\pi v}}\,e^{-\frac{1}{2v}(2m-w)^2} \tag{15}$$

## The Joint Density Function of a Brownian Motion Without Drift and Its Minimum

We start by taking the derivative of the left side of Equation (6) with respect to $w$ which is...

$$\frac{\delta}{\delta w}\bigg(\int\limits_{-\infty}^{m}\int\limits_{w}^{\infty} g(x,y)\,\delta y\,\delta x\bigg) = -\int\limits_{-\infty}^{m} g(x,w)\,\delta x \tag{16}$$

We then take the derivative of Equation (16) with respect to $m$ which is...

$$\frac{\delta}{\delta m}\bigg(\int\limits_{-\infty}^{m} g(x,w)\,\delta x\bigg) = -g(m,w) \tag{17}$$

We now move to the right side of Equation (6) and take its derivatives with respect to $m$ and $w$. Using the definition of theta in Equation (7) the right side of Equation (6) can be rewritten as...

$$F = \int\limits_{-\infty}^{\theta} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}z^2}\,\delta z \tag{18}$$

The derivative of Equation (18) with respect to $m$ is...

$$\frac{\delta F}{\delta \theta}\frac{\delta \theta}{\delta m} = \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}\theta^2} \times 2 = \frac{2}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(2m-w)^2} \tag{19}$$

We now take the derivative of Equation (19) with respect to $w$. Using the definition of lambda in Equation (8) Equation (19) can be rewritten as...

$$G = \frac{2}{\sqrt{2\pi v}}\, e^{\lambda} \tag{20}$$

The derivative of Equation (20) with respect to $w$ is...

$$\frac{\delta G}{\delta \lambda}\frac{\delta \lambda}{\delta w} = \frac{2}{\sqrt{2\pi v}}\, e^{\lambda} \times \frac{2m-w}{v} = \frac{2(2m-w)}{v\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(2m-w)^2} \tag{21}$$

Equation (17) and Equation (21) are the derivatives of the left and right side, respectively, of Equation (3) with respect to the variables $m$ and $w$. When we reverse the signs and equate these two equations the equation for the joint density of a Brownian motion with zero drift and its minimum is...

$$g(m,w) = -\frac{2(2m-w)}{v\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(2m-w)^2} = \frac{2(w-2m)}{v\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(2m-w)^2} \tag{22}$$

## Adding Drift to the Brownian Motion $W_T$

The $w$ in the joint density functions as defined by Equations (15) and (22) have zero drift and variance $v$. We want to multiply this equation by the multiplier $\xi(w)$ such that $w$ now has drift equal to $\alpha$ and variance equal to $v$. By using the following multipler we can move the mean (i.e. drift) of the probability distribution but keep its variance unchanged...

$$\xi(w) = e^{\frac{\alpha w}{v} - \frac{\alpha^2}{2v}} \tag{23}$$

See the PDF 'Moving The Mean Of A Normal Distribution'.

We will define a new joint density function $\hat{f}(m,w)$ to be the product of the joint density function in Equation (15), which has zero drift, and the multiplier in Equation (23). The equation for the new joint density function $\hat{f}(m,w)$ is...

$$\begin{aligned}\hat{f}(m,w) &= f(m,w)\,\xi(w)\\ &= \frac{2(2m-w)}{v\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(2m-w)^2}\, e^{\frac{\alpha w}{v}-\frac{\alpha^2}{2v}}\\ &= \frac{2(2m-w)}{v\sqrt{2\pi v}}\, e^{\frac{\alpha w}{v}-\frac{\alpha^2}{2v}-\frac{1}{2v}(2m-w)^2}\end{aligned} \tag{24}$$

We will define a new joint density function $\hat{g}(m,w)$ to be the product of the joint density function in Equation (22), which has zero drift, and the multiplier in Equation (23). The equation for the new joint density function $\hat{g}(m,w)$ is...

$$\begin{aligned}\hat{g}(m,w) &= g(m,w)\,\xi(w) \\ &= \frac{2(w-2m)}{v\sqrt{2\pi v}}\,e^{-\frac{1}{2v}(2m-w)^2}\,e^{\frac{\alpha w}{v}-\frac{\alpha^2}{2v}} \\ &= \frac{2(w-2m)}{v\sqrt{2\pi v}}\,e^{\frac{\alpha w}{v}-\frac{\alpha^2}{2v}-\frac{1}{2v}(2m-w)^2}\end{aligned} \tag{25}$$

## Some Hypothetical Problems

The probability distribution that uses the joint density functions as defined by Equations (24) and (25) should integrate to one given that every $w$ and $m$ pair is represented by the double integral that contains those two functions. Note that in the problems below $w$ represents the value of the Brownian motion at time $t$ and $m$ represents the maximum or minimum of the Brownian motion over the interval [0, $t$].

**Problem 1**: Define the double integral that uses Equation (24) such that it represents all possible pairs of the Brownian motion and its maximum and therefore integrates to one.

**Answer**:

$$\int_{w=-\infty}^{w=0}\int_{m=0}^{m=\infty}\hat{f}(m,w)\,\delta m\,\delta w+\int_{w=0}^{w=\infty}\int_{m=w}^{m=\infty}\hat{f}(m,w)\,\delta m\,\delta w=1.0000 \tag{26}$$

**Problem 2**: Define the double integral that uses Equation (25) such that it represents all possible pairs of the Brownian motion and its minimum and therefore integrates to one.

**Answer**:

$$\int_{w=-\infty}^{w=0}\int_{m=-\infty}^{m=w}\hat{g}(m,w)\,\delta m\,\delta w+\int_{w=0}^{w=\infty}\int_{m=-\infty}^{m=0}\hat{g}(m,w)\,\delta m\,\delta w=1.0000 \tag{27}$$

**Problem 3** - What is the probability that a Brownian motion with zero drift and variance $v = t$ crosses the barrier $m = 0.75$ (maximum) sometime before time $t$ and ends up below $w = 0.25$ at time $t$ assuming that $t = 2.00$? (This is Problem 1 from Part I). What is the solution using the joint density function in Equation (15) above?

**Answer**:

$$\begin{aligned}Prob\Big[M_t^+ > 0.75, W_t < 0.25\Big] &= \int_{w=-\infty}^{w=0.25}\int_{m=0.75}^{m=\infty} f(m,w)\,\delta m\,\delta w \\ &= \int_{w=-\infty}^{w=0.25}\int_{m=0.75}^{m=\infty}\frac{2(2m-w)}{v\sqrt{2\pi v}}\,e^{-\frac{1}{2v}(2m-w)^2}\,\delta m\,\delta w \\ &= 0.18838\end{aligned} \tag{28}$$

**Problem 4** - What is the probability that a Brownian motion with drift equal to $\alpha = \mu t$ and variance equal to $v = \sigma^2 t$ crosses the barrier $m = -0.25$ (minimum) sometime before time t and ends up above $w = -0.05$ at time $t$ assuming that $t = 2.00$, $\mu = 0.10$ and $\sigma = 0.80$? (This is Problem 2 from Part I but with non-zero drift). What is the solution using the joint density function in Equation (25) above?

**Answer**:

$$\begin{aligned}Prob\Big[M_t^- < -0.25, W_t > -0.05\Big] &= \int_{w=-0.05}^{w=\infty}\int_{m=-\infty}^{m=-0.25}\hat{g}(m,w)\,\delta m\,\delta w \\ &= \int_{w=-0.05}^{w=\infty}\int_{m=-\infty}^{m=-0.25}\frac{2(w-2m)}{v\sqrt{2\pi v}}\,e^{\frac{\alpha w}{v}-\frac{\alpha^2}{2v}-\frac{1}{2v}(2m-w)^2}\,\delta m\,\delta w \\ &= 0.3821\end{aligned} \tag{29}$$